CHAPTER

13 Straight Lines

- **Distance Formula:** $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
- **Section Formula:** $x = \dfrac{mx_2 \pm nx_1}{m \pm n}$; $y = \dfrac{my_2 \pm ny_1}{m \pm n}$
- **Centroid, Incentre & Excentre:**

Centroid $G\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$,

Incentre $I\left(\dfrac{ax_1 + bx_2 + cx_3}{a+b+c}, \dfrac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

Excentre $I_1\left(\dfrac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \dfrac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$

Remarks:

(*i*) If the triangle is equilateral, then centroid, incentre, orthocenter, circumcenter, coincides.

(*ii*) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining. Orthocentre and circumcentre in the ratio 2 : 1.

(*ii*) In a isosceles triangle centroid, orthocentre, incentre, circumcentre lies on the same line.

Area of Triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left\| \frac{1}{2} \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \right\|$$

Equation of Straight Line

(*a*) Equation of *a* line parallel to *x*-axis at *a* distance a is $y = a$ or $y = -a$.

(*b*) Equation of *x*-axis is $y = 0$.

(*c*) Equation of line parallel to *y*-axis at a distance *b* is $x = b$ or $x = -b$.

(*d*) Equation of *y*-axis is $x = 0$.

If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line $AB = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Standard Forms of Equations of a Straight Line

(*a*) **Slope Intercept form :** Let *m* be the slope of a line and *c* its intercept on *y*-axis, then the equation of this straight line is written as : $y = mx + c$.

(*b*) **Point Slope form :** If m be the slope of a line and it passes through a point (x_1, y_1), then its equation is written as : $y - y_1 = m(x - x_1)$.

(*c*) **Two point form :** Equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is written as :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(*d*) **Intercept form :** If *a* and *b* are the intercepts made by a line on the axes of *x* and *y*, its equation is written as : $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

(*e*) **Normal form :** If *p* is the length of perpendicular on a line from the origin and α the angle which this perpendicular makes with positive *x*-axis, then the equation of this line is written as :

$x \cos \alpha + y \sin \alpha = p$ (*p* is always positive), where $0 \leq \alpha < 2\pi$.

(*f*) **Parametric form :** $\dfrac{x - h}{\cos \theta} = \dfrac{y - k}{\sin \theta} = r$ is the equation.

(*g*) **General form :** We know that a first degree equation in *x* and *y*, $ax + by + c = 0$ always represents a straight line. This form is known as general form of straight line.

(*i*) Slope of this line $= \dfrac{-a}{b} = \dfrac{\text{coefficient of } x}{\text{coefficient of } y}$

(*ii*) Intercept by this line on x-axis $= -\dfrac{c}{a}$ and intercept by this line on *y*-axis $= -\dfrac{c}{b}$.

(*iii*) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2+b^2}$.

Angle Between Two Lines

(*a*) If θ be the angle between two lines : $y = m_1x + c_1$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$.

(*b*) If equation of lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then these line are–

(*i*) Parallel $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(*ii*) Perpendicular $\Leftrightarrow a_1a_2 + b_1b_2 = 0$

(*iii*) Coincident $\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(*iv*) Intersecting $\Leftrightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Length of Perpendicular from a Point on a Line

Length of perpendicular from a point (x_1, y_1) on the line $ax+by+c=0$ is $= \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$.

In particular the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \frac{|c|}{\sqrt{a^2+b^2}}$.

Distance Between two Parallel Lines

(*i*) The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$.

(**Note :** The coefficients of x & y in both equations should be same).

(*ii*) The area of the parallelogram $= \frac{p_1p_2}{\sin\theta}$, where p_1 & p_2 are distance between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

Equation of lines Parallel and Perpendicular to a Given Line

(*i*) Equation of line parallel to line $ax + by + c = 0$.

$$ax + by + \lambda = 0$$

(*ii*) Equation of line perpendicular to line $ax + by + c = 0$.

$$bx - ay + k = 0$$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

Straight Line Making a given Angle with a Line

Equations of lines passing through a poing (x_1, y_1) and making an angle α, with the line $y = mx + c$ is written as :

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Position of Two Points with Respect to a Given Line

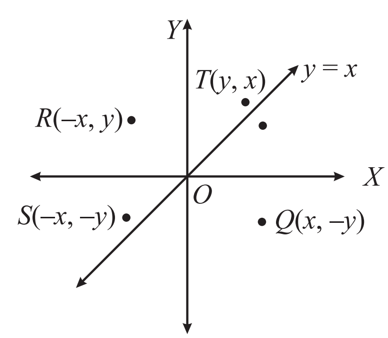
Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

Concurrency of Lines

Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent, if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

Reflection of a Point

Let $P(x, y)$ be any point, then its image with respect to



(*i*) x-axis is $Q(x, -y)$

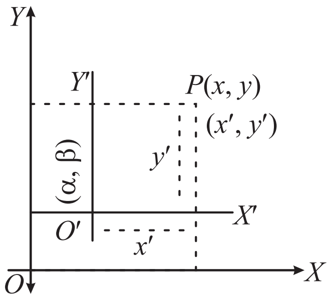
(*ii*) y-axis is $R(-x, y)$

(*iii*) origin is $S(-x, -y)$

(*iv*) line $y = x$ is $T(y, x)$

Transformation of Axes

(*a*) **Shifting of origin without rotation of axes :** If coordinates of any point $P(x, y)$ with respect to new origin (a, b) will be (x', y')

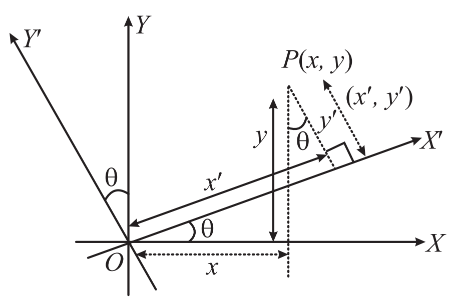


then $x = x' + \alpha, \quad y = y' + \beta$

or $x' = x - \alpha, \quad y' = y - \beta$

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting $x + \alpha$ in place of x and $y + \beta$ in place of y.

(*b*) **Rotation of axes without shifting the origin :** Let O be the origin. Let $P \equiv (x, y)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes OX' and OY', where $\angle X'OX = \angle YOY' = \theta$



then $x = x' \cos\theta - y' \sin\theta$

$\quad y = x' \cos\theta - y' \cos\theta$

or $y' = x \cos\theta + y \sin\theta$

$\quad y' = -x \sin\theta + y \cos\theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

New \ Old	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

Equation of Bisectors of Angles between Two Lines

If equation of two intersecting lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then equation of bisectors of the angles between these lines are written are:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad \text{...(1)}$$

(*a*) **Equation of bisector of angle containing origin :** If the equation of the lines are written with constant terms c_1 and c_2 positive, then the equation of bisectors of the angle containing the origin is obtained by taking sign in (1).

(*b*) **Equation of bisector of acute/obtuse angles :** See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by –1 to make the constant terms positive.

Determinate the sign of $a_1a_2 + b_1b_2$

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sing in eq. (1)	use – sign in eq. (1)
–	use – sign in eq. (1)	use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Family of Lines

If equation of two lines be $P = a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x + b_2y + c_2 = 0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional information given in the problem.

General Equation and Homogeneous Equation of Second Degree

(*a*) A general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of straight lines if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

(*b*) If θ be the angle between the lines, then $\tan\theta = \pm \dfrac{2\sqrt{h^2 - ab}}{a + b}$.

Obviously these lines are

(*i*) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$.

(*ii*) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(*c*) Homogeneous equation of 2nd degree $ax^2 + 2hxy + by^2 = 0$ always represent a pair of straight lines whose equations are

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right) x \equiv y = m_1x \text{ and } y = m_2x$$

and $\quad m_1 + m_2 = -\dfrac{2h}{b}\,;\ m_1m_2 = \dfrac{a}{b}$

These straight lines passes through the origin and for finding the angle between these lines same formula as given for general equation is used.

The condition that these lines are :

(*i*) At right angles to each other is $a + b = 0$. i.e. co-efficient of x^2 + co-efficient of $y^2 = 0$.

(ii) Coincident is $h^2 = ab$.

(iii) Equally inclined to the axis of x is $h = 0$. i.e. coefficient of $xy = 0$.

(d) The combined equation of angle bisectors between the lines represented by homogeneous equation of 2nd degree is given by

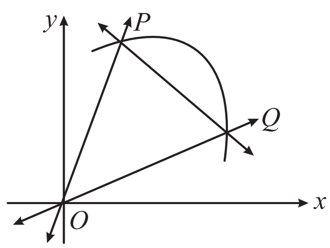
$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}, a \neq b, h \neq 0.$$

(e) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

(f) If lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel then distance between them is $= 2\sqrt{\dfrac{g^2 - ac}{a(a+b)}}$.

Equations of Lines Joining the Points of Intersection of a Line and a Curve to the Origin

Let the equation of curve be:



$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

and straight line be

$lx + my + n = 0$ ...(ii)

$$ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0$$

STANDARD RESULTS

(i) Area of rhombus formed by lines $a|x| + b|y| + c = 0$ or $\pm ax \pm by + c = 0$ is $\dfrac{2c^2}{|ab|}$.

(ii) Area of triangle formed by line $ax + by + c = 0$ and axes is $\dfrac{c^2}{2|ab|}$.

(iii) Co-ordinate of foot of perpendicular (h, k) from (x_1, y_1) to the line $ax + by + c = 0$ is given by $\dfrac{h - x_1}{a} = \dfrac{k - y_1}{b} = \dfrac{-(ax_1 + by_1 + c)}{a^2 + b^2}$.

(iv) Image of point (x_1, y_1) w.r. to the line $ax + by + c = 0$ is given by

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}.$$

Chart

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\Delta = 0$ → Pair of straight line

- → $h^2 = ab, g^2 = ac, f^2 = bc$ (Pair of coincident line)
- → $h^2 = ab, g^2 \neq ac, f^2 \neq bc$ (Pair of coincident line)
- → $h^2 = \neq ab$ (Pair of straight line not passing through origin)

(ii) $ax^2 + 2hxy + by^2 = 0$

- $h^2 = ab$ → Pair of coincident lines passing through origin
- $a + b = 0$ → Pair of perpendicular lines passing through origin
- $h^2 \neq ab$ → Pair of lines passing through origin